CSc 545: Homework Assignment 1

Assigned: Monday, August 22 2016
Due: 9:30 AM, Monday, Sept 5 2016

Clear, neat and concise solutions are required in order to receive full credit so revise your work carefully before submission, and consider how your work is presented. If you cannot solve a particular problem, state this clearly in your write-up, and write down only what you know to be correct. For complicated proofs, first outline the argument and the delve into the details.

1. (10 pts) Prove or disprove:
   (a) $2^{n+1} = O(2^n)$
   (b) $2^{2n} = O(2^n)$
   (c) $(\log n)^{\log n} = \Theta(n^{\log \log n})$
   (d) $(n + 1)^2 = O(n^2)$.
   (e) $cn^k + d = O(2^n)$, for all $c, d, k \in R^+$

2. (20 pts) Graph problems:
   (a) Prove by induction that a graph with $n$ vertices can have at most $n(n - 1)/2$ edges.
   (b) Prove by induction that $V - E + F = 2$, where $V$ is the number of vertices, $E$ is the number of edges and $F$ is the number of faces of a graph.
   (c) Prove that a planar graph has at most $3n - 6$ edges.
   (d) Prove that an outerplanar graph has at most $2n - 3$ edges, where a graph is outerplanar if it is planar and it can be drawn such that all vertices lie on the outer face of the drawing.

3. (30 pts) Geometry problems:
   (a) Prove that any set of regions defined by $n$ lines in the plane can be colored with two colors so that no two regions that share an edge have the same color.
   (b) Consider a set of straight-line segments in the plane with the following property: any two segments are either disjoint, or one touches the other in an interior point (two segments cannot touch with each of them using an endpoint). Given $n$ such segments, what is the maximum number of touching pairs? Provide a formal proof.
   (c) Replace “straight-line segments” in the problem above with “simple curve” and answer the same question. Provide a formal proof.

4. (20 pts) As an employee of Citibank’s investment division you are asked to compare the performance of stock brokers in the company. In order to do this, you are asked to compare the broker’s performance to the optimum possible. To do this, you first need to solve the following problem: Consider the performance of a given stock over $n$ consecutive days. You need to find out the maximum amount of money a broker could have made by buying when the stock was the cheapest and selling when the stock was worth the most. The question is on what day should the stock have been bought and on what day it should have been sold in order to make as much money as possible. If there was no way to make money during the $n$ days in question, you should report this instead.
   (a) Design and analyze an algorithm that solves this problem in $O(n^2)$ time.
   (b) Design and analyze an algorithm that solves this problem in $O(n \log n)$ time.
5. (20 pts) In the same Citibank problem, consider the problem of the “happy banker.” A banker is happy if his stock investment does not lose value. Thus, once a banker buys a stock, he would be happiest if every day the stock value is worth no less than what is was the day before. Sometimes this is not possible; then the banker is happiest when his stock doesn’t lose its value for the maximum number of days possible. Now Citibank wants to evaluate the happiness of its stock brokers, against the maximum possible. To do this, you first need to solve the following problem: Consider the performance of a given stock over \( n \) consecutive days. You need to find the maximum number of days, such that the value of the stock on the selected day was no less than that of the previous selected (not necessarily consecutive) day.

   (a) Design and analyze a \( O(n^2) \) algorithm for this problem.

   (b) Design and analyze a \( O(n \lg n) \) dynamic programming algorithm for this problem.

Extra Credit: During your latest visit to Mt. Lemmon, you noticed a long icicle, hanging off the ski lift. Enthralled by the powder, and possibly injured by the multiple face-plants, you forgot about it until the last run. You noticed that it had fallen down, broken into 3 pieces which formed a nice-looking isosceles triangle. On your way down the Catalina highway, you’ve tried to calculate the probability that the three pieces form a triangle (not just isosceles, but any triangle). 45 minutes later, you still were not sure what the right answer is. So, what is it?