Recurrence Relations & Recursion

Computer Science has recursion.

Mathematics has recurrence relations.

Example(s):
Recurrence Relations

**Definition:** Recurrence Relation

Example(s):

Solving Recurrence Relations
Linear Homogeneous Recurrence Relations

Definition: Linear Homogeneous Recurrence Relation
With Constant Coefficients (LHRRWCC) Of Degree $k$

Example(s):

Solving LHRRWCCs Of Degree 2 (1 / 2)
Theorem: Assume a characteristic equation
\[ w^2 - c_1w - c_2 = 0 \] with \( c_1, c_2 \in \mathbb{R} \) and roots \( r_1 \) and \( r_2 \) such that \( r_1 \neq r_2 \). The sequence \( \{R(n)\} \) is a solution to \( R(n) = c_1R(n-1) + c_2R(n-2) \) iff
\[ R(n) = \alpha_1r_1^n + \alpha_2r_2^n \] where \( n \in \mathbb{Z}^* \) and \( \alpha_1, \alpha_2 \in \mathbb{R} \).

Solution Procedure: LHRRWCCs of Degree 2

1. Identify \( c_1 \) & \( c_2 \) and create the characteristic equation
   \[ w^2 - c_1w - c_2 = 0 \]

2. Insert the roots of the characteristic equation \( (r_1 \& r_2) \)
   into the closed-form expression \( R(n) = \alpha_1r_1^n + \alpha_2r_2^n \)

3. Using the initial conditions, create two equations in two unknowns \( (\alpha_1 \text{ and } \alpha_2) \)

4. Solve for \( \alpha_1 \) and \( \alpha_2 \) to complete the solution
Example: Solving a LHRRWCC of Degree 2

Solve: \( R(n) = 3R(n - 1) - 2R(n - 2) \)
where \( R(0) = 200 \) and \( R(1) = 220 \).

“Divide & Conquer” Recurrence Relations (1 / 2)

Background:

From the Latin *Divide Et Impera* ("divide and rule")
Solving Divide & Conquer Rec. Relations (1 / 6)

“Find The Pattern” (a.k.a. Iterative (or Backward) Substitutions)

Example(s):
Conjecture: \( S(n) = k \cdot \log_2 n + 1 \)
Solving Divide & Conquer Rec. Relations (4 / 6)

Example(s):

Solving Divide & Conquer Rec. Relations (5 / 6)
Solving Divide & Conquer Rec. Relations (6 / 6)

**Conjecture:** \( Q(n) = \frac{n(n+1)}{2} \)

Approximate Solutions to Rec. Relations (1 / 2)

**Theorem:** (The Master Theorem) Given a recursive function of the form \( T(n) = a \cdot T(n/b) + c \cdot n^d \), where:

- \( T(n) \) is an increasing function,
- \( n = b^k \),
- \( k \) is an integer \( > 0 \),
- \( a \) is a real \( \geq 1 \),
- \( b \) is an integer \( > 1 \),
- \( c \) is a real \( > 0 \), and
- \( d \) is a real \( \geq 0 \), then:

\[
 f(n) = \begin{cases} 
 O(n^d) & \text{if } a < b^d \\
 O(n^d \cdot \log_2 n) & \text{if } a = b^d \\
 O(n^{\log_b a}) & \text{if } a > b^d 
\end{cases}
\]

**Proof:** Rosen 7/e, Exercises 29-33 of Section 8.3.
Example(s):

Binary Search’s recurrence: \( S(n) = S\left(\frac{n}{2}\right) + k \)

Recall: We determined \( S(n) = k \cdot \log_2 n + 1 \Rightarrow O(\log_2 n) \)

From the Master Theorem: \( T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n^d \)

For Bin. Search, \( a = 1, \quad b = 2, \quad c = k, \quad \text{and} \quad d = 0 \)

The 2nd case applies: \( a = b^d \quad (1 = 2^0) \)

Therefore, \( S(n) \) is \( O(n^d \cdot \log_2 n) \), or \( O(\log_2 n) \).

⇒ We got it right!

Note: Master Theorem doesn’t fit Quicksort’s worst case.