Propositions with Variables (1 / 2)

Propositions are static; variables are not allowed. But...

Definition: Predicate (a.k.a. Propositional Function)

Example(s):
Definition: Domain (a.k.a. Universe) of Discourse

Example(s):
Quantification

Idea: Establish truth of predicates over sets of values.

Two common generalizations:

Note: Do not use the book’s non-standard $\exists!x$ notation.
1. Universally Quantified Predicates

Example(s):
2. Existentially Quantified Predicates

Example(s):
First: Distinguishing $\exists x \forall y S(x, y)$ from $\forall i \exists k T(i, k)$:
Example(s):
Consider this conversational English statement:

All of McCann’s students are geniuses.

How can we express that statement in logic notation?
Example: Universal Quantification (2 / 5)

Attempt #2: All of McCann’s students are geniuses. → Logic
Example: Universal Quantification (3/5)

Attempt #3: All of McCann’s students are geniuses. → Logic

Let $P(x)$: Student $x$ is a genius, $x \in \text{People}$
Example: Universal Quantification (4 / 5)

Attempt #4: All of McCann’s students are geniuses. → Logic

Let $P(x) : \text{Student } x \text{ is a genius, } x \in \text{People}$

Let $M(x) : x \text{ is enrolled in one of McCann’s classes, } x \in \text{People}$
Example: Universal Quantification (5 / 5)

Attempt #5: All of McCann’s students are geniuses. → Logic

Let $P(x) : x$ is a genius, $x \in \text{People}$

Let $M(x) : x$ is enrolled in one of McCann’s classes, $x \in \text{People}$
Implicit Quantification

The “all” can be implicit in the English statement.

Example(s):
Example: Existential Quantification

Consider this conversational English statement:

At least one towel is dirty.

How can we express that statement in logic notation?
Express this more specific statement in logic:

Some of the blue guest towels are dirty.

Let $D(x) : x \text{ is dirty, } x \in \text{Towels}$
Yet Another Example: Quantification

Now express this statement in logic:

Every last one of the blue guest towels are dirty!

Let $B(x) : x$ is blue, $x \in$ Towels
Let $G(x) : x$ is used by guests, $x \in$ Towels
Let $D(x) : x$ is dirty, $x \in$ Towels
Free vs. Bound Variables

Definition: Bound Variable

Definition: Free (a.k.a. Unbound) Variable

Other examples of ‘binding’ in CS:
Negations of Quantified Expressions

Remember De Morgan’s Laws for propositions? Well, . . .

Definition: Generalized De Morgan’s Laws
Demonstration: \( \forall x P(x) \equiv \exists x P(x) \) (1 / 2)

Let \( S(x) : x < 4, x \in \mathbb{Z} \)

The expression \( \forall x S(x), x \in \{1, 2, 3\} \) is true.

Equivalently, \( \forall x S(x) \) is false.

\[
\forall x S(x) \equiv S(1) \land S(2) \land S(3) \quad \text{so \ldots}
\]

\[
\overline{\forall x S(x)} \equiv \overline{S(1) \land S(2) \land S(3)}
\]

\[
\equiv \overline{S(1)} \lor \overline{S(2)} \lor \overline{S(3)} \quad \text{[De Morgan, 2x]}
\]

(Remember: \( \overline{S(1)} \lor \overline{S(2)} \lor \overline{S(3)} \) is still false.)

(Continues \ldots)
Demonstration: \( \forall x \, P(x) \equiv \exists x \, P(x) \) (2 / 2)

For \( S(1) \lor S(2) \lor S(3) \) to be false, each term must be false; that is, no \( S(x) \) is true (or all \( S(x) \) are false).

It follows that the expression \( \exists x \, S(x) \) must be false, completing the demonstration.

Example(s):
Consider this conversational (& correct!) English statement:

Only one citizen of North Dakota is a member of the U.S. House of Representatives.

And consider this awkward but useful rewording:
That rewording is useful because it can be directly expressed logically:
Expressing “Exactly two . . .” Statements (1 / 3)

Key observation:

**Question:** Can you say this in ‘awkward English’?

Exactly two citizens of North Dakota are U.S. Senators.
Consider the two halves separately:

1. “At least two citizens of North Dakota are U.S. Senators”

2. “At most two citizens of North Dakota are U.S. Senators”
Finally, AND together

\[ \exists x \exists y \left( S(x) \land S(y) \land (x \neq y) \right) \]

and

\[ \forall x \forall y \forall z \left( (S(x) \land S(y) \land S(z)) \rightarrow (x = y \lor y = z \lor x = z) \right) : \]