Collected Definitions for Exam #3

**Topic 9: Functions**

- A *function* from set $X$ to set $Y$, denoted $f : X \rightarrow Y$, is a relation from $X$ to $Y$. If $(x, y) \in f$, then $y$ is the only value returned from $f(x)$. Further, $f(x)$ is defined $\forall x \in X$.

- For each of the following, let $f : X \rightarrow Y$ be a function, and assume $f(n) = p$.
  - $X$ is the *domain* of $f$.
  - $Y$ is the *codomain* of $f$.
  - $f$ maps $X$ to $Y$.
  - $p$ is the *image* of $n$.
  - $n$ is the *pre-image* of $p$.
  - The *range* of $f$ is the set of all images of elements of $X$. (Note that the range need not equal the codomain.)

- The *floor* of $n$, denoted $\lfloor n \rfloor$, is the largest integer $\leq n$.

- The *ceiling* of a value $m$, denoted $\lceil m \rceil$, is the smallest integer $\geq m$.

- A function $f : X \rightarrow Y$ is *injective* (a.k.a. *one-to-one*) if, for each $y \in Y$, $f(x) = y$ for at most one member of $X$.

- A function $f : X \rightarrow Y$ is *surjective* (a.k.a. *onto*) if $f$’s range is $Y$ (the range = the codomain).

- A *bijective* function (a.k.a. a *one-to-one correspondence*) is both injective and surjective.

- The *inverse* of a bijective function $f$, denoted $f^{-1}$, is the relation $\{(y, x) \mid (x, y) \in f\}$.

- Let $f : Y \rightarrow Z$ and $g : X \rightarrow Y$. The *composition* of $f$ and $g$, denoted $f \circ g$, is the function $h = f(g(x))$, where $h : X \rightarrow Z$.

- A function $f : X \times Y \rightarrow Z$ (or $f(x, y) = z$) is a *binary* function.

**Topic 10: Properties of Integers**

- Let $i$ and $j$ be positive integers. $j$ is a *factor* of $i$ when $i \% j = 0$.

- A positive integer $p$ is *prime* if $p \geq 2$ and the only factors of $p$ are 1 and $p$.

- A positive integer $p$ is *composite* if $p \geq 2$ and $p$ is not prime.

- Let $x$ and $y$ be integers such that $x \neq 0$ and $y \neq 0$. The *Greatest Common Divisor* (GCD) of $x$ and $y$ is the largest integer $i$ such that $i \mid x$ and $i \mid y$. That is, $\gcd(x, y) = i$.

- If the GCD of $a$ and $b$ is 1, then $a$ and $b$ are *relatively prime*.

- When the members of a set of integers are all relatively prime to one another, they are *pairwise relatively prime*.

- Let $x$ and $y$ be positive integers. The *Least Common Multiple* (LCM) of $x$ and $y$ is the smallest integer $s$ such that $x \mid s$ and $y \mid s$. That is, $\text{lcm}(x, y) = s$. 

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Topic 11: Sequences and Strings

- A sequence is the ordered range of a function from a set of integers to a set S.
- In an arithmetic sequence (a.k.a. arithmetic progression) \(a, a_{n+1} - a_n\) is constant. This constant is called the common difference of the sequence.
- In a geometric sequence (a.k.a. geometric progression) \(g, \frac{a_{n+1}}{g_n}\) is constant. This constant is called the common ratio of the sequence.
- An increasing (a.k.a. non-decreasing) sequence \(i\) is ordered such that \(i_n \leq i_{n+1}\).
- A strictly increasing sequence \(i\) is ordered such that \(i_n < i_{n+1}\).
- A non-increasing (a.k.a. decreasing) sequence \(i\) is ordered such that \(i_n \geq i_{n+1}\).
- A strictly decreasing sequence \(i\) is ordered such that \(i_n > i_{n+1}\).
- Sequence \(x\) is a subsequence of sequence \(y\) when the elements of \(x\) are found within \(y\) in the same relative order.
- A string is a contiguous finite sequence of zero or more elements drawn from a set called the alphabet.
- A set is finite if there exists a bijective mapping between it and a set of cardinality \(n, n \in \mathbb{Z}^*\).
- A set is countably infinite (a.k.a. denumerably infinite) if there exists a bijective mapping between the set and either \(\mathbb{Z}^*\) or \(\mathbb{Z}^+\).
- A set is countable if it is either finite or countably infinite. If neither, the set is uncountable.

Topic 12: Induction

- The First Principle of Mathematical Induction: if (i) \(P(a)\) is true for the starting point \(a \in \mathbb{Z}^+\), and (ii) if \(P(k)\) is true for any \(k \in \mathbb{Z}^+\), then \(P(k+1)\) is true, then \(P(n)\) is true for all \(n \in \mathbb{Z}^+, n \geq a\).
- The Second Principle of Mathematical Induction: if (i) \(P(a)\) is true for the starting point \(a \in \mathbb{Z}^+\), and (ii) (for any \(k \in \mathbb{Z}^+\)) if \(P(j)\) is true for any \(j \in \mathbb{Z}^+\) such that \(a \leq j \leq k\), then \(P(k+1)\) is true, then \(P(n)\) is true for all \(n \in \mathbb{Z}^+, n \geq a\).

Topic 13: Counting

- I provided two definitions of the (Generalized) Pigeonhole Principle; learn either one (or both!):
  (a) if \(n\) items are placed in \(k\) boxes, then at least one box contains at least \(\lceil \frac{n}{k} \rceil\) items.
  (b) Let \(f : X \to Y\), where \(|X| = n\) and \(|Y| = k\), and let \(m = \lceil \frac{n}{k} \rceil\). There are at least \(m\) values \((a_1, a_2, \ldots, a_m)\) such that \(f(a_1) = f(a_2) = \ldots = f(a_m)\).
- The Multiplication Principle (a.k.a. the Product Rule): If there are \(s\) steps in an activity, with \(n_1\) ways of accomplishing the first step, \(n_2\) of accomplishing the second, etc., and \(n_s\) ways of accomplishing the last step, then there are \(n_1 \cdot n_2 \cdot \ldots \cdot n_s\) ways to complete all \(s\) steps.
- The Addition Principle (a.k.a. the Sum Rule): If there are \(t\) tasks, with \(n_1\) ways of accomplishing the first, \(n_2\) ways of accomplishing the second, etc., and \(n_t\) ways of accomplishing the last, then there are \(n_1 + n_2 + \ldots + n_t\) ways to complete one of these tasks, assuming that no two tasks interfere with one another.
- The Principle of Inclusion-Exclusion for Two Sets says that the cardinality of the union of sets \(M\) and \(N\) is the sum of their individual cardinalities excluding the cardinality of their intersection. That is: \(|M \cup N| = |M| + |N| - |M \cap N|\)
- The Principle of Inclusion-Exclusion for Three Sets says that the cardinality of the union of sets \(M, N,\) and \(O\) is the sum of their individual cardinalities excluding the sum of the cardinalities of their pairwise intersections and including the cardinality of their intersection. That is: \(|M \cup N \cup O| = |M| + |N| + |O| - (|M \cap N| + |M \cap O| + |N \cap O|) + |M \cap N \cap O|\)